

CASSON–GORDON INVARIANTS OF SOME 3-FOLD BRANCHED COVERS OF KNOTS

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Received 16 February 1987

Revised 19 February 1988

In this paper the Casson–Gordon invariants of certain knots are computed using threefold branched covers. These knots provide examples of $(4n+3)$ -dimensional doubly sliced knots which are not the double of a disk. Examples are also given of $(4n+3)$ -dimensional knots which are algebraically doubly sliced but not geometrically doubly sliced.

AMS (MOS) Subj. Class.: 57Q45, 57Q60

doubly sliced knots Casson–Gordon invariants
double disk knots

Introduction

A knot is doubly sliced if it can be realized as an equatorial slice of the unknot. Levine [6] showed that double disk knots, knots formed by unioning two identical disk knots along their boundary, are doubly sliced. Simple odd dimensional doubly sliced knots are all double disk knots.

In [8] the author used the Casson–Gordon invariants to detect examples of $(4n+1)$ -dimensional knots that are doubly sliced but not the double of a disk knot. In this paper we show that the knots described in [8] also yield examples of $(4n+3)$ -dimensional doubly sliced knots which are not doubled disks. The technique used to detect these knots is the same as in [8] but applied to 3-fold branched cyclic covers. The Casson–Gordon invariants of the $(4n+3)$ -knots constructed in [8] and [7] vanish for the 2-fold covers but not on the 3-fold covers. The additional difficulty in using higher degree covers is the added complexity in computing the Atiyah–Singer α -invariants.

This method also shows that some of the $(4n+3)$ -knots constructed by Ruberman in his thesis [7] are algebraically doubly sliced but not geometrically doubly sliced. A knot is algebraically doubly sliced if it satisfies the Sumners–Levine condition. Ruberman shows in [7] that the constructed $(4n+1)$ -knots are algebraically doubly sliced but not doubly sliced. Ruberman’s $(4n+3)$ -dimensional examples were

originally also believed to have this property. However, an error was found. He has recently found new $(4n+3)$ -dimensional examples using Wall's realization of L -groups.

The author wishes to thank Danny Ruberman and Jerry Levine for many helpful discussions.

1. Definitions

An n -knot is a smooth oriented pair (S^{n+2}, K) where K is homeomorphic to S^n . The knot K is null-cobordant if there is an $(n+1)$ -disk knot (B^{n+3}, Δ) , i.e., Δ is a properly imbedded $(n+1)$ -disk in the $(n+3)$ -ball B^{n+3} , and $\partial(B, \Delta) = (S, K)$. The knot K is doubly sliced if K has two null cobordisms (B_1, Δ_1) and (B_2, Δ_2) such that $(B_1, \Delta_1) \cup -(B_2, \Delta_2)$ is the $(n+1)$ -unknot. Necessary conditions for odd dimensional knots to be double sliced were found by Sumners [10] and extended by Levine [6]. These are sufficient for simple knots. Knots which satisfy Sumners' and Levine's conditions are called algebraically doubly sliced. A subclass of doubly sliced knots was found by Levine [6]: these are the double disk knots. A knot (S^{n+2}, K) is a double disk knot if $(S^{n+2}, K) = (B^{n+2}, \Delta) \cup_I -(B^{n+2}, \Delta)$.

Levine used surgery to show that double disk knots are doubly sliced but one can see this relationship explicitly. It is a theorem of Zeeman that the 1-twist spin of any knot K is trivial [11]. If K is the double of (B^{n+2}, Δ) and (S^{n+3}, K_t) is the 1-twist spin of K then we argue that K is a slice of K_t . The equator of K_t is $K \# -K$ which contains a copy of (B^{n+2}, Δ) . A neighborhood of this (B^{n+2}, Δ) is $(B^{n+2}, \Delta) \times I$ which has (S^{n+2}, K) as its boundary (see Fig. 1).

In order to detect our examples we employ two theorems which are stated below. These theorems rely on the Casson–Gordon invariants which we define after the theorems.

Theorem 1 [8]. *Let (S^{2q+1}, K) be a double disk knot and N a branched cyclic cover of (S^{2q+1}, K) . Then there is a direct sum decomposition $H_1(N) = A \oplus B$ satisfying*

- (i) *there is an epimorphism $A \twoheadrightarrow B$,*
- (ii) *if $\phi : H_1(N) \rightarrow Z_d$ is a map such that $\phi|_B = 0$ then $\sigma(N, \phi) = 0$.*

Theorem 2 [7]. *Let (S^{2q+1}, K) be a doubly sliced knot and N a branched cyclic cover of (S^{2q+1}, K) . Then there is a decomposition $H_1(N) = A \oplus B$ satisfying:*

- if $\phi : H_1(N) \rightarrow Z_d$ ($d = p^r$, prime power) with $\phi|_A = 0$*
- or $\phi|_B = 0$ then $|\sigma(N, \phi)| \leq \dim H_q(N, Z_p)$.*

Let M be a $(2k-1)$ -manifold and d an integer. Isomorphism classes of Z_d -covers of M with a specified generator for the covering translations correspond to elements of $[M, BZ_d] = H^1(M, Z_d) = \text{Hom}(H_1(M), Z_d)$. If $\psi \in \text{Hom}(H_1(M), Z_d)$ we get

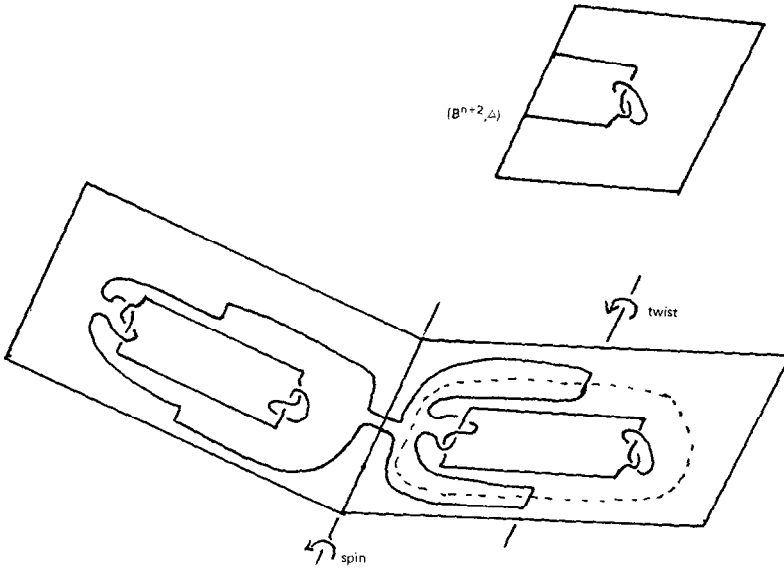


Fig. 1.

(\tilde{M}, τ) , such a covering with a generator. There exists an n and a $2k$ -manifold with boundary W such that $\langle \tilde{W}, T \rangle$ is a Z_d cover of W and $\partial \langle \tilde{W}, T \rangle = n \cdot \langle \tilde{M}, \tau \rangle$,

$$\partial \begin{pmatrix} \tilde{W} \\ \downarrow \\ W \end{pmatrix} = n \begin{pmatrix} \tilde{M} \\ \downarrow \\ M \end{pmatrix}.$$

This fact is essentially that $\Omega_{2k-1}(BZ_d)$ is torsion. Let $\bar{H}_k(W, \psi) = e^{2\pi i/d}$ eigenspace of T_* in $H_k(\tilde{W}) \otimes \mathbb{C}$. If k is even then the intersection form, $\langle x, y \rangle = x \cdot y$, is Hermitian and if k is odd the form $\langle x, y \rangle = ix \cdot y$ is Hermitian. Let $\bar{\sigma}(X, \psi)$ denote the signature of $\langle \cdot, \cdot \rangle|_{\bar{H}_k(W, \psi)}$. We define the Casson-Gordon invariant as

$$\sigma(M, \psi) = \frac{1}{n} (\bar{\sigma}(W, \psi) - \sigma(W)).$$

Gilmer [4] shows this invariant to be well-defined.

An m component link of dimension n or an m -link is an ordered collection of m disjoint smooth oriented submanifolds of S^{n+2} , each of which is homeomorphic to S^n . We assume $n > 2$ and our links will be ordered. A link is denoted by L or $(S^{n+2}; L_1, \dots, L_m)$. Every m -link is bounded by a Seifert surface, W . A link is a boundary link if it has an m -component Seifert surface $W = W_1 \cup \dots \cup W_m$ with $\partial W_i = L_i$.

Let $V_1, \dots, V_m \subset D^{n+3}$ (or S^{n+2}) be disjoint codimension two submanifolds with trivial normal bundles. We require that $V_i \cap \partial D^{n+3} = \partial V_i = L_i$ (or $\partial V_i = \emptyset$ in S^{n+2}). We call such a set $(D^{n+3}; V_1, \dots, V_m)$ or $(S^{n+2}; V_1, \dots, V_m)$ a special m -tuple.

Given a special m -tuple we can construct a branched $Z_{a_1} \oplus \cdots \oplus Z_{a_m}$ cover of D^{n+3} or S^{n+3} , M_V . The multisignatures of M_V are a link invariant and if L is a boundary link of dimension $2q-1$ with matrix $S = (A_{ij})$, A_{ij} is an $l_i \times l_j$ matrix (see [5]), then the multisignatures are given by

$$\text{sig}_L(\omega_1, \dots, \omega_m) = \begin{cases} \text{sign}(i(I-W)(-SW^{-1}-S^T)) & q \text{ even,} \\ \text{sign}((I-W)(-SW^{-1}+S^T)) & q \text{ odd,} \end{cases}$$

where

$$W = \begin{bmatrix} [\omega_m] & & & 0 \\ & [\omega_2] & & \\ & & \ddots & \\ 0 & & & [\omega_m] \end{bmatrix},$$

$[\omega_i] = \omega_i I_{l_i \times l_i}$ and ω_i is an a_i th root of unity.

For more details see [8, 9].

If J is a knot, $L = L_1 \cup L_2$ is a link and r, r_1, r_2 are integers then [7] and [8] describe knots associated to J and L denoted $K(J, r)$ and $K(L; r_1, r_2)$ respectively (see Fig. 2). These knots bound an $(n+1)$ -manifold in S^{n+2} which consists of n -handles (either J or L) and 1-handles. The importance of this construction is that

- (i) $K(J, r)$ and $K(L; r_1, r_2)$ are algebraically doubly sliced.
- (ii) If L_1 and L_2 are sliced as knots then $K(L; r, s)$ is doubly sliced.

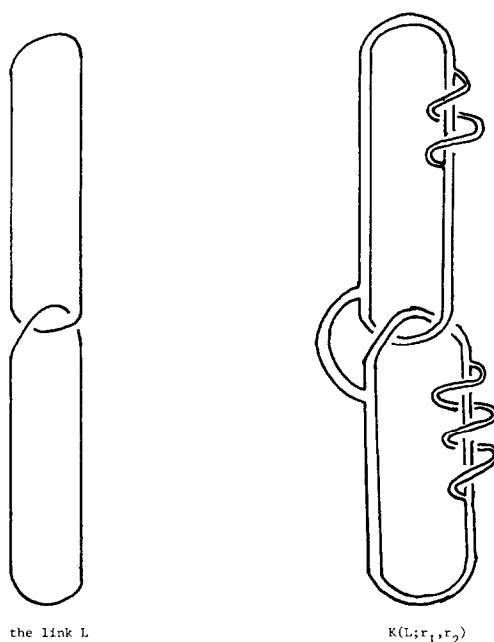


Fig. 2.

2. 3-Fold branched covers

We wish to describe the 3-fold branched cyclic cover of S^{n+2} over $K(L; r_1, r_2)$ and $K(J, r)$. The 2-fold covers are constructed in [7] and [8] so we only sketch the differences. Let N_K denote the 3-fold cover of $K(L; r_1, r_2)$ (or $K(J, r)$). Then $N_K = \partial W$ where W is the 3-fold cover of B^{n+3} branched over a Seifert surface for K with its interior pushed into the interior of B^{n+3} . A handlebody construction of W is essentially given in [1] with necessary modifications described in [7], [8] and [9]. We obtain a surgery on a link description of N_K (Fig. 3). Here the link $l_1 \cup l_2 \cup l'_1 \cup l'_2$ is formed by taking connected sums of the components from two copies of L and a copy of $L \parallel L$, L union of a parallel push off of L .

We now compute $H_1(N_K)$. Let μ_i be the meridian to l_i and μ'_i the meridian to l'_i , where the l_i and l'_i are the attaching regions for the $n+1$ handles. Denote the attaching regions of the 2-handles by s_i and s'_i . The linking between the attaching regions of the 2-handles and the $(n+1)$ -handles is as follows:

- s_i links l_i $2r_i + 1$ times
- s_i links l'_i r_i times
- s'_i links l_i $r_i + 1$ times
- s'_i links l'_i $2r_i + 1$ times for $i = 1, 2$.

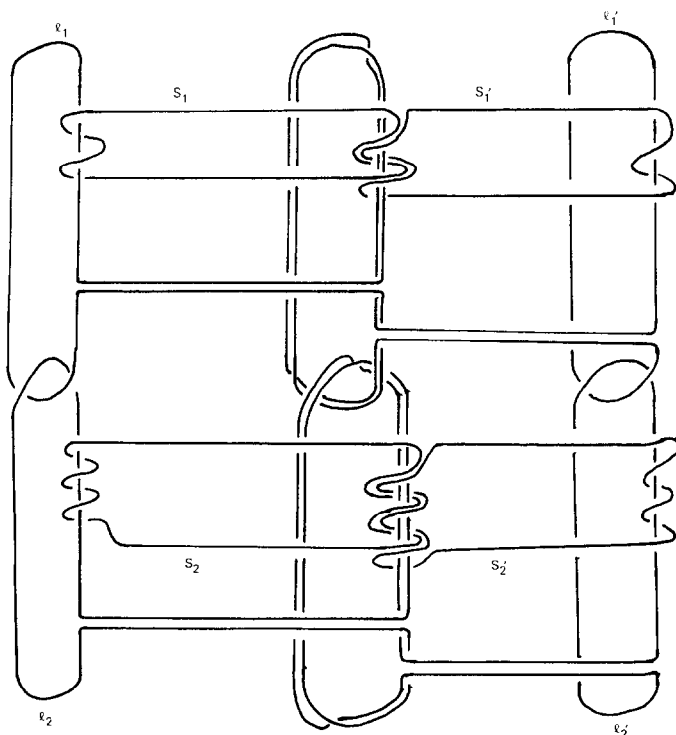


Fig. 3. "Surgery on a link" description of $N_{K(L; r_1, r_2)}$.

In order to compute $H_1(N_K)$ we first do surgery along the l_i and l'_i which yields a manifold with first homology $Z \oplus Z \oplus Z \oplus Z$. The effect of performing surgery along the s_i and the s'_i is given by

$$0 \rightarrow Z^4 \xrightarrow{M} Z^4 \xrightarrow{\gamma} H_1(N_K) \rightarrow 0$$

$$\begin{bmatrix} 2r_1+1 & r_1+1 & & 0 \\ r_1 & 2r_1+1 & & \\ & & 2r_2+1 & r_2+1 \\ 0 & & r_2 & 2r_2+1 \end{bmatrix} = M$$

where

$$\mu_1 = \gamma \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mu'_1 = \gamma \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_2 = \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mu'_2 = \gamma \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

By performing a change in basis on each group, we can determine that

$$H_1(N_K) = Z_{3r_1^2+3r_1+1} \oplus Z_{3r_2^2+3r_2+1}$$

and the relation between the μ 's, which is

$$\mu_1 = (-(3r_1+2), 0), \quad \mu'_1 = (1, 0), \quad \mu_2 = (0, -(3r_2+2)) \quad \text{and} \quad \mu'_2 = (0, 1).$$

This calculation is summed up in the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z^4 & \xrightarrow{M} & Z^4 & \xrightarrow{\gamma} & H_1(N_K) \longrightarrow 0 \\ & & \downarrow M_1 & & \downarrow M_2 & & \downarrow \\ 0 & \longrightarrow & Z^4 & \xrightarrow{M_3} & Z^4 & \xrightarrow{\rho} & H_1(N_K) \longrightarrow 0 \end{array}$$

$$M_1 = \begin{bmatrix} 2r_1+1 & r_1+1 & & 0 \\ 2 & 1 & & \\ & & 2r_2+1 & r_2+1 \\ 0 & & 2 & 1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 1 & 0 \\ -(3r_1+2) & 1 \\ & 1 & 0 \\ & -(3r_2+2) & 1 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(3r_1^2+3r_1+1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(3r_2^2+3r_2+1) \end{bmatrix}.$$

3. Casson–Gordon invariants

Suppose we have a representation of $Z_{a_1} \oplus \cdots \oplus Z_{a_m}$ into $GL(V)$ with V a complex vector space. If T_i is the generator of Z_{a_i} , then we can simultaneously diagonalize the T_i 's and decompose V into $\oplus E_V(\omega_1, \dots, \omega_m)$ where $E_V(\omega_1, \dots, \omega_m) = \bigcap_{i=1}^m (\omega_i\text{-eigenspace of } T_i)$.

If $\phi: H_1(N^{2q+1}) \rightarrow Z_d$ is realized by the covering space $\tilde{N} \rightarrow N$ then $\sigma(N, \phi)$ may be computed using a bounded Z_d -manifold \hat{M} where $\hat{M}/Z_d = M$ and $\partial \hat{M} \rightarrow \partial M$ is the cover $\tilde{N} \rightarrow N$. An argument similar to that in [3] yields

$$\sigma(N, \phi) = \text{sign}|_{E_{H_{q+1}(M; \mathbb{C})}(\omega)} \langle \cdot, \cdot \rangle - \sigma(M) - F(\hat{M}, Z_d).$$

The formula $F(\hat{M}, Z_d)$ arises from the G -signature theorem [2] and involves data about the fixed points and their normal bundles. We show how to compute $\sigma(N_K, \phi)$ for all $\sigma: H_1(N_K) \rightarrow Z_d$ where $K = K(L; r_1, r_2)$.

Lemma 3. Suppose $G = \bigoplus_{j=1}^m Z_d$ acts on M and let $p: G \twoheadrightarrow Z_d$ be given by $p(\prod T_j^{x_j}) = T^{\sum_j x_j c_j}$ for $c_1, \dots, c_m \in \mathbb{Z}$. Further, let $H = \text{kernel } p$ and suppose that M/H is a manifold, then

$$E_{H^*(M/H; \mathbb{C})}(\omega^i) \xrightarrow{\rho} E_{H^*(M; \mathbb{C})}(\omega^{ic_1}, \omega^{ic_2}, \dots, \omega^{ic_m})$$

is an isomorphism. Here ω is the d th root of unity $e^{2\pi i/d}$.

Proof. By the transfer homomorphism we know that

$$\begin{array}{ccc} H^*(M/H; \mathbb{C}) & \xrightarrow{\rho^*} & (H^*(M; \mathbb{C}))^H \\ & & \downarrow \rho \\ & & M/H \end{array}$$

is an equivariant isomorphism.

If $v \in E_{H^*(M/H; \mathbb{C})}(\omega^i)$ then

$$T_j \rho^*(v) = \rho^* T_j^c v = \omega^{ic_j} \rho^* v \quad \text{for all } j.$$

If $x \in E_{H^*(M; \mathbb{C})}(\omega^{ic_1}, \dots, \omega^{ic_m})$ then x is fixed by H so $x = \rho^* v$ for $v \in H^*(M/H; \mathbb{C})$. Now, $\rho^*(Tv) = \prod T_j^{a_j} \rho^* v$ where $\sum a_j c_j = 1 \pmod d$, so

$$\rho^* Tv = \prod_j \omega^{ic_j a_j} \rho^* v = (\omega^{\sum c_j a_j})^i \rho^* v = \omega^i \rho^* v. \quad \square$$

Lemma 4. Let M_V be the canonical $Z_d \oplus \cdots \oplus Z_d$ -manifold associated to $(S^{2q+2}; V_1, \dots, V_m)$ then

$$\text{sign}|_{E_{H^{q+1}(M_V; \mathbb{C})}(\omega^{i_1, \omega^{a_2}, \dots, \omega^{a_m}})} \langle \cdot, \cdot \rangle = \frac{-2^{2q}}{d} \sum_{j=1}^m \sigma(V_j)(d - 2ic_j) + \sigma(M).$$

The proof is a computation similar to Lemma 5 in [8] and Lemma 2.1 in [3].

Theorem 5. Let $L = L_1 \cup \cdots \cup L_m \subset S^{2q+1}$ be an m -link and A_{jj} a Seifert matrix of the knot L_j . Let $W = B^{2q+2} \cup \{h_i^2\} \cup \{h_j^{2q}\}$ where the h_i^2 are 2-handles and the h_j^{2q} are $2q$ -handles attached along L_j . Also let $M = \partial W$ and $\mu_j \in H_1(M)$ be represented by the j th meridian. If the map $\phi : H_1 M \rightarrow Z_d$ given by $\phi(\mu_j) = c_j$ is a well defined homomorphism then

$$\sigma(M, \phi) = \text{sig}_L(\omega^{c_1}, \dots, \omega^{c_m}) + \frac{2^{2q}}{d} \sum (d - 2c_j) \sigma(A_{jj} + A_{jj}^T).$$

Proof. Let V_j be the Seifert surface to L_j with its interior pushed into the interior of B^{2q+2} . As in [7] Theorem 3.5 we can calculate $\sigma(M, \phi)$ using M_V/H where M_V is the canonical $Z_d \oplus \cdots \oplus Z_d$ -manifold associated to $(B^{2q+2}; V)$ and H is the kernel of $\tilde{\phi}$ where $\tilde{\phi}$ is given by

$$\begin{array}{ccc} H_1(X) & \xrightarrow{\quad} & Z_d \\ \downarrow & \searrow \tilde{\phi} & \\ Z_d \oplus Z_d \oplus \cdots \oplus Z_d & & \\ \nearrow H & & \\ 0 & & \end{array}$$

$X = \text{link complement,}$
 $\phi(x_1, \dots, x_n) = \sum_{i=1}^m c_i x_i.$

Applying the two lemmas yields the result. \square

Examples

If S is a Seifert matrix for a boundary link $L = L_1 \cup L_2$ then the link $l = l_1 \cup l_2 \cup l'_1 \cup l'_2$ derived from L in Section 2 has Seifert matrix

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & S & S & 0 \\ 0 & -S^T & -S^T & 0 \\ 0 & 0 & 0 & -S^T \end{bmatrix}.$$

If the components of L are all knots with signature zero then $F(\hat{M}, Z_d) = 0$ and the Casson–Gordon invariants of $K(L; r, s)$ from its 3-fold branched cover are

$$\begin{aligned} \sigma &= \text{sig}_l(\omega^i, \omega^j, \omega^{-(3r+2)i}, \omega^{-(3s+2)j}) \\ &= \text{sig}_L(\omega^i \omega^j) - \text{sig}_L(\omega^{-(3r+2)i}, \omega^{-(3s+2)j}) \\ &\quad + \text{sig}_{L \parallel L}(\omega^i, \omega^j, \omega^{-(3r+2)i}, \omega^{-(3s+2)j}). \end{aligned}$$

From [8], we know that if $d = p \cdot q$, p and q distinct primes, and $K(L; r, s)$ is a double disk knot then all the Casson–Gordon invariants must vanish.

Let L be the $(4q+3)$ -dimensional simple 2-link given by the following matrix of matrices (see [5]):

$$\begin{bmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{bmatrix}.$$

The diagonal sub-matrices specify the knot type of each component. The components of L are unknotted so $K(L; 1, 4)$ is a doubly sliced knot and $H_1(N_K) = Z_7 + Z_{61}$. We therefore take $Z_d = Z_{427} = Z_7 + Z_{61}$. If $i = 61$ and $j = 140$ so that $\omega^i = e^{2\pi i/7}$ and $\omega^j = e^{(2\pi i/61) \cdot 20}$, then $\sigma = -2$ so $K(L; 1, 4)$ is not a doubled disk.

If J^{4q+3} is the simple knot given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then J has signature 0 and $K(J, 1)$ is algebraically doubly sliced. By Ruberman's theorem, if d is prime and $K(J, 1)$ is doubly sliced then all its Casson–Gordon invariants must vanish. The Casson–Gordon invariants of $K(J, 1)$ arising from its 3-fold cover are

$$\sigma = \text{sig}_J(\omega^i) - \text{sig}_J(\omega^{-(3r+2)i}) + \text{sig}_{J \parallel J}(\omega^i, \omega^{-(3r+2)i}).$$

Since $H_1(M_{K(J,1)}) = Z_7$ if we take $d = 7$ and if $i = 1$ then $\sigma = +2$ so $K(J, 1)$ is not doubly sliced.

References

- [1] S. Akbulut and R. Kirby, Branched covers of surfaces in 4-manifolds, *Math. Ann.* 252 (1980) 111–132.
- [2] M.F. Atiyah and I.M. Singer, The index of elliptic operators: III, *Ann. Math.* 87 (1968) 546–604.
- [3] A. Casson and C. Gordon, On slice knots in dimension three, in: *Proc. Symp. Pure Math.* 32 (1978).
- [4] P. Gilmer, Configurations of surfaces in 4-manifolds, *Trans. AMS* 264 (1981) 353–380.
- [5] Ki Hyoung Ko, Seifert matrices and boundary link cobordisms, *Trans. AMS* 299 (1987) 657–681.
- [6] J. Levine, Doubly sliced knots and doubled disk knot, *Michigan Math. J.* 30 (1983) 249–256.
- [7] D. Ruberman, Doubly sliced knots and the Casson–Gordon invariants, *Trans. AMS* 279 (1983) 569–588.
- [8] L. Smolinsky, Doubly sliced knots which are not the double of a disk, *Trans. AMS* 298 (1986) 723–732.
- [9] L. Smolinsky, A generalization of the Levine–Tristram link invariant, *Trans. AMS*, to appear.
- [10] D.W. Sumners, Invertible knot cobordisms, *Comment. Math. Helv.* 46 (1971) 240–256.
- [11] E.C. Zeeman, Twisting spun knots, *Trans. AMS* 115 (1965) 471–495.